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ABSTRACT

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Elementary Subjects Center
Series No. 91

TEACHING AND LEARNING MATHEMATICS FOR UNDERSTANDING IN A FIFTH-GRADE CLASSROOM

Ralph T. Putnam and James W. Reineke

Center for the Learning and Teaching of Elementary Subjects

Institute for Research on Teaching
College of Education
Michigan State University

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Abstract

This report considers what it means for one teacher, Elaine Hugo, to teach mathematics for understanding. By looking closely at the goals Hugo has for her students and how these goals are reflected in how she talks about her teaching and what she does in the classroom, the authors explore what teaching for understanding means for her. The authors also look closely at a two-week unit on decimal fractions. Prior to and after this unit six target students were interviewed about their knowledge of decimals. In spite of Hugo's extensive knowledge of mathematics and concern that her students understand the mathematics they study, only two of the six target students showed different ways of thinking about decimals after the decimals unit. The authors conclude that teaching for understanding is a complex process. This complexity arises from many questions teachers trying to teach for understanding must address. These questions include what content should be taught in elementary mathematics classes, what it means to understand something, and what instructional techniques should teachers use.

TEACHING AND LEARNING MATHEMATICS FOR UNDERSTANDING IN A FIFTH-GRADE CLASSROOM

Ralph T. Putnam and James W. Reineke[1]

This report considers what it means for one teacher, Elaine Hugo,[2] to teach mathematics for understanding. Hugo teaches fourth- and fifth-grade mathematics half time, spending the other half of her time as the district's mathematics coordinator (this is her first year in this role). Hugo is committed to helping her students learn mathematics that goes beyond the computational skills so emphasized in traditional curriculum and instruction and to empower students to enjoy and learn mathematics as they progress through school.

We will paint a portrait of Hugo's mathematics teaching first by considering what she says in interviews about her goals for students--what she thinks is important that students learn about mathematics. These goals are in many ways consistent with reform documents like the National Council of Teachers of Mathematics Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989). Then we fill in this sketch by examining in more detail what it means for Hugo to teach mathematics for understanding, exploring both how she talks about her teaching and what she actually does in the classroom.

Data for this analysis come from a variety of sources. We interviewed Hugo about her goals for students and her teaching practices. We observed the fifth-grade mathematics class one or two days each week for the entire 1989-90 school year, relying on audiotape and written fieldnotes to record our observations. We often had brief conversations with Hugo before or after class to bring us up to date on what had happened on days that we were not observing. We also interviewed students in Hugo's class about the mathematics they were learning and their

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[2]Names of teacher and students are pseudonyms. The research reported here was conducted in an elementary school in a midsize urban town in mid-Michigan. The school population was made up of students from various ethnic and social backgrounds.

beliefs and attitudes about mathematics. We conducted more intensive data collection for a two-week period during which Hugo taught a unit on decimal fractions. We videotaped all of the lessons during this two-week period and conducted interviews with six target students about their knowledge of decimals.

Hugo's Goals for Students' Mathematics Learning

In a December interview, we asked Hugo about what she wants her students to learn and how she thinks about teaching mathematics. We asked her to talk about the role of elementary school in helping students learn mathematics, about her goals for her fifth-grade students, and how she thought about teaching mathematics. In this interview, Hugo sketched out an overall image of mathematics teaching and learning that is consistent with the broad visions of mathematics teaching and learning painted in reform documents such as the NCTM (1989) Curriculum Standards.

Just as reformers want students to "learn to value mathematics" (NCTM, 1989), Hugo's first goal for her students is[3] that they come to enjoy mathematics and want to learn more of it: "When my kids leave and go on to sixth grade I hope first of all that they have . . . a good attitude about mathematics." Similarly, when talking about the role of elementary school in general, Hugo began by saying,

> Ideally to help that child, you know the child that comes in is the kindergartner who is so full of life and enthusiastic for learning. Ideally to be able to keep that going. To help that child to learn how to learn, whether it is math, whether it is science or whether it is reading or even getting along with other people. . . . There are going to be so many things that are different in life than we have right now and I think that it is going to be more important for these kids maybe than a lot in the past to be able to continue learning on their own.

Hugo sees these goals of a positive attitude toward mathematics and the desire to learn more as being especially important for girls:

> I feel that if they have that going into the middle school it is at least going to keep their mind open as far as trying to learn more mathematics. Especially for the girls. So many girls at this stage start to tune out of math-type of things and

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[3]In describing Hugo's goals and beliefs, we use present tense, recognizing that these beliefs may have changed since 1989.

> possibly cut themselves off from career opportunities, so an attitude of enjoying mathematics [is important].

This emphasis on girls, who have traditionally avoided mathematics as they progress through school, is also consistent with national groups' emphasis on the need to make mathematics more accessible for traditionally disenfranchised groups, such as women and minorities (NCTM, 1989; National Research Council, 1989).

Also like the reformers, Hugo places an emphasis on understanding over learning isolated computational skills. Equally important to students enjoying mathematics is her goal that students gain a strong foundational understanding of arithmetic. Hugo said she wants her fifth graders to gain

> a basic understanding of addition, subtraction, multiplication, and division. Some way of algorithm, some algorithm that they are comfortable with. Whether it is the traditional division algorithm or whether it's an offshoot. If they can find the answer of a division problem and I am not talking three digit by six digit kind of thing. . . . They need to have a sense of whole number [algorithms], what they mean and how to use them. And then some beginnings into fractions, decimals, probability, statistics, geometry.

Computational skills, according to Hugo, should be taught with an emphasis on understanding, application, and problem solving. When we asked about the relationship between computational skill and understanding, Hugo argued that if you teach for conceptual understanding, students will learn and remember the needed computational skills:

> If they really understand it, I think the computation is going to come right along with it. Whereas, kids can follow the rules, kind of memorize and pass the test and then forget about it. . . . So if we can get the kids to understand the concepts and get the computation through understanding the concepts then they are going to be able to remember it. . . . I mean kids can . . . whether it is in sixth, seventh, or eighth grade, and they have learned a rule, they can tell me how to multiply fractions, but they have no idea of what it means or when they would use it. So if we teach for the concept and they can do the computation, I think it is going to stick with them, whereas, if we teach for the computation, we are going to have to keep teaching it.

When we asked Hugo about the role of problem solving, she said:

> Ideally, problems are the way that I would like to see mathematics taught instead of presenting, well fractions might be an example to use. Instead of presenting how to multiply fractions out of context, how to divide fractions--if anybody cares--out of context, . . . if we could begin by presenting problems that need that particular skill so that the kids have some idea why they need to learn how to multiply fractions. I'm not sure they need to learn how to divide fractions but at

least if we can try to give them some examples of where they are going to ever use that or at least of what it means.

Thus, as Hugo talked with us about what she wanted for her students, it became clear that her goals are in important ways consistent with the NCTM's (1989) Curriculum Standards and other reform documents. She wants to instill an enjoyment of mathematics and a desire to learn more, as well as an understanding of arithmetic concepts and when to use them. She wants students to learn arithmetic with understanding, not through rote memorization and drill.

But "understanding" can mean different things to different people. A variety of teaching practices and more specific beliefs about teaching and learning could be consistent with the images sketched out in reform documents like the NCTM (1989) Curriculum Standards and with the fairly general goals that Hugo described. By observing Hugo's mathematics teaching and talking with her over the year we came to understand more fully Hugo's teaching and the kinds of changes she was trying to make. We learned more about what it means for Hugo to teach for understanding and how her teaching plays out in the classroom.

The remainder of this report considers Hugo's teaching in more detail, both how she talked about it and what she did in the classroom. We draw on our interviews with Hugo and our classroom observations, as well as trying to situate Hugo's teaching and beliefs in the context of the more formalized positions of the research literature. To provide a context for examining Hugo's teaching, we will first sketch out a single lesson in her classroom. The lesson illustrates several salient features of Hugo's teaching and her interactions with students. Like many of Hugo's lessons, it included both whole-class and small-group work. The lesson also served as the focus of a stimulated-recall interview, in which we viewed a videotape of the lesson together with Hugo and talked about what she was thinking and trying to do in the lesson.

Overview of a Lesson

The lesson began with Hugo having students show various decimal numbers with base-ten blocks on place-value mats (shown in Figure 1). Hugo had introduced representing decimal fractions with base-ten blocks in a previous lesson, along with the convention that a flat

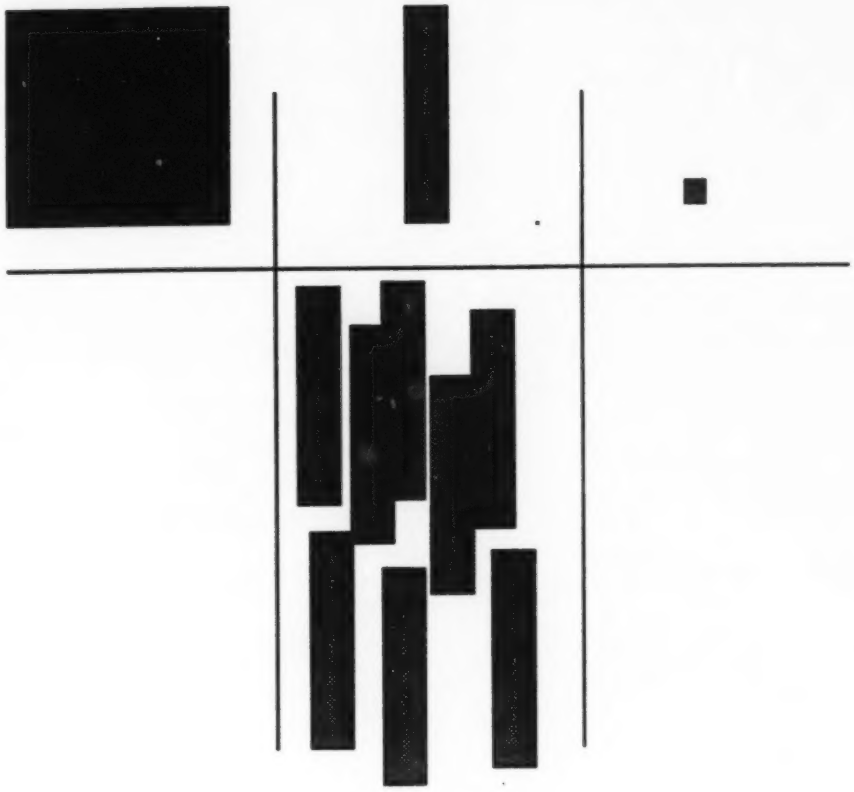

Figure 1. Place-value mat and base-ten blocks representing .8.

(usually used to show 100) represents 1. Each student had a mat and base-ten blocks on which to represent the decimals that Hugo announced. Hugo had individual students come up to the overhead to show the correct way to display the decimals and she monitored individual students' representations with the blocks. The decimals included tenths, hundredths, and wholes, and discussion focused on how to represent various decimals and what kinds of exchanges could be made (e.g., 10 tenths [longs] being the same as 1 whole [flat]).

After about 10 minutes of this activity, Hugo wrote the words "decimals" and "fractions" on the board to begin a discussion about similarities between these kinds of numbers. During the stimulated-recall interview, Hugo said that her primary goal here was to get students to talk about the idea that both decimals and fractions involve working with parts of wholes and that decimals always involve dividing a whole up into 10 or some number of parts that is a power of 10. A number of other ideas, however, came up and were explored during the discussion. These included discussion of whether 2 was a fraction or a decimal; whether decimals can be negative numbers and, if so, how they might be represented with base-ten blocks; and whether any decimal can be converted into a fraction. Because the idea of both fractions and decimals involving parts of wholes was not brought up by the students, Hugo raised this issue by using pictures of circles divided into eighths and tenths and getting the students to talk about them. The idea that decimals always involve dividing wholes into powers of 10 followed from this discussion. The entire discussion lasted about 20 minutes.

Hugo next introduced a task that students were to work on in their "learning-buddy" groups--heterogeneous groups of three or four students in which particular roles, such as timekeeper and recorder, are assigned. The task for the groups was a worksheet on which they were to use base-ten blocks to represent a variety of decimal numbers. Members of each group were to sign the jointly completed worksheet to indicate that they understood all the solutions on it. The students worked in their groups for about 15 minutes, at which point they brought their completed papers to the teacher to check.

Hugo spent the final 10 minutes before recess commenting on the groups' success at working together and discussing ways that one of the groups might have been more efficient in assigning group roles. In discussing how the groups did in terms of communicating with one another and completing the task jointly, Hugo pointed out that later in the week they would be playing a game in which each individual in the group would compete against individuals from other groups[4] and therefore it was important that all individuals in the group understood the material. She also discussed a difficulty that several of the groups seemed to be having with the tasks on the worksheet--confusion over writing numbers as tenths using decimal points or words. For example, one of the problems on the worksheet should have been answered "5 tenths," but several students wrote ".5 tenths" instead. At 10:00 students went out for recess.

When students returned about 15 minutes later, Hugo spent the remaining 15 minutes of math class introducing a decimals game that students would be playing in learning-buddy groups the next day. The game involved randomly selecting cards with decimal numbers on them (e.g., 0.25) and shading that amount in on paper divided into 100 squares. We draw on this lesson throughout the remainder of the report. For convenience we will refer to it as "the showing decimals lesson."

Teaching for Understanding: A Closer Look

As we got to know Elaine Hugo and her classroom we were struck with how committed she was to improving mathematics teaching--both in her own classroom and in other classrooms in her district. Like many committed teachers, she was continually trying to improve. Partly because she was serving half time as the district's mathematics coordinator, Hugo actively sought out and examined a large variety of innovative curricular materials, intended both for students and teachers. (Her primary task this year as mathematics coordinator was to work with a group of teachers and a university consultant to come up with a set of district

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[4]This was apparently a version of one of Slavin's (1983) cooperative learning game structures that Hugo had learned about in district-sponsored inservice workshops on cooperative learning.

expectations for mathematics education in district elementary schools and review options for replacing or supplementing the existing curriculum.)

Because she actively sought out new ideas in her quest to improve, Hugo came into contact with a number of different materials and ideas that influenced her teaching. She had been to inservice workshops on cooperative learning and the district-required training in effective teaching. She watched videotapes demonstrating how to use various manipulatives in teaching. She pored over supplementary curricular materials touted as promoting understanding and problem solving as well as the district's adopted text, Comprehensive School Mathematics Program [CSMP](CEMREL, 1985). She eagerly sought input from these various sources to provide new activities or techniques that she could use to improve her own teaching and help other teachers improve as well.

Sometimes these various sources harbored assumptions and perspectives that complemented one another; in other cases, they were based on what seemed--at least to us as outsiders--as competing assumptions. But Hugo worked hard to bring together ideas and techniques from these diverse recommendations into a pedagogy that worked for her. The result is that Hugo's teaching and the way she talks about it, like most teachers' was an amalgam--eclectic and ever-changing.

Because teachers, like all learners, interpret information and experiences through their existing knowledge and beliefs, however, taking a closer look at how Hugo incorporated various ideas and techniques into her teaching helped up us better understand the knowledge and beliefs that shaped her teaching. As we spent more time in Hugo's classroom over the year, a number of consistencies emerged, revealing beliefs about what is important for students to learn about mathematics and implicit beliefs about the teaching and learning process.

What is Understanding?

Surfacing again and again in our conversations with Hugo and observations in her classroom were indicators of what she meant when she said she wanted students to understand the mathematics they were learning. When we asked Hugo during the December interview what

she meant when she said it was important to develop a foundational understanding of arithmetic, she emphasized the need for students to be able to explain what basic facts or algorithms mean and to be able to use their knowledge in other situations. A student who understood basic facts, for example,

> could tell me that 7 times 8 means that you have 7 groups of 8 or 8 groups of 7, or they can give me an example of um . . . 7 of my friends each have 8 cookies, how many cookies do I have? [There is some] total kind of situation that they could verbalize and explain and give and example and then from there use that.

About understanding place value, Hugo said:

> To know and be able to tell me, like 23. What is the value of the 2? What is the value of the 3? The 2 is really worth 20 not 2. That it is worth two groups of 10. Being able to understand the place value, the ones, the tens, the hundreds, and possibly into the decimals part. . . . To be able to understand those as far as the tenths, the hundredths, etc., but also to be able to use that information, for instance, when they are multiplying. If they are multiplying 24 times 36 and they come up with answer that is just in the 100s, to have a sense, you know, the number sense that, the place-value understanding to know that their answer is not feasible. It is not accurate, it is not close. So not just to be able to tell me that it is two groups of 10 but to again, be able to use it in context and to be able to help them check their work.

In describing what it would mean to understand multiplying fractions, Hugo similarly emphasized being able to provide an explanation:

> Well, an idea of that if I gave them a fraction, a multiplication problem like 1/3 times 2/5. They could . . . they could come up with maybe a picture of what that really meant and/or a story line maybe along with it. Or that they could draw a picture . . . you know that goes back to the whole idea that understanding that a fraction really relates to the whole. Whatever the whole is--and the whole might be different if we are talking about pizzas or classroom sizes or whatever. But they could take a picture and show me 2/5 of something and then find 1/3 of that by drawing that 2/5 into three equal parts. So I guess if they could put it into a picture form, I would be happy that they had a pretty good idea of what it really meant.

For Hugo, understanding means being able to explain, perhaps by drawing a picture, why a computational procedure works and having the ability to use the procedure or algorithm in a problem or real-life situation. This emphasis on usefulness is consistent with the current reform's emphasis on students' having accessible knowledge that they can use in a variety of situations (NCTM, 1989).

The traditional topics of the school mathematics curriculum--addition, subtraction, multiplication, and division with whole numbers, decimals and fractions--provide the core of the curriculum for Hugo, although she wants students to learn these traditional topics with understanding, rather than as rote computational procedures. Part of this emphasis on the computational topics of arithmetic comes from Hugo's concern that her students do well in the computationally oriented middle school they will attend after they leave her fifth-grade classroom. Upon entering middle school, students are given a computation test, which is used to place them in mathematics class. Hugo feels obligated to ensure her students can do as well as possible on this test, even if she does not particularly value the importance of computational skills for their own sake.

Representations Play a Central Role

When it comes to helping students acquire these understandings, Hugo works to provide carefully structured lessons and activities that will lead students to the desired understandings. She thinks carefully about how to teach particular mathematical topics, typically considering how the topic is treated in the district curriculum (CSMP) and in a variety of other sources. An important part of this planning is making decisions about what concrete materials or visual representations to use to teach particular concepts.

Hugo's thinking about representations is more systematic than that of many teachers and that evidenced in most traditional mathematics textbooks. In commenting on the use of manipulatives in teaching elementary mathematics, Ball (1991a, 1992) described four approaches. The first approach emphasizes letting students "play" or explore with manipulatives, without making explicit connections to mathematical ideas or symbols. A second approach seems to assume that concrete materials can directly model mathematical truth—that as long as students manipulate physical materials as a part of instruction, they will inevitably come to correct mathematical understandings. Ball argues that both of these approaches simplistically assume that having students use concrete materials will automatically result in greater student understanding of mathematics.

In a third approach, Ball argues, the teacher acknowledges the ambiguities in the concrete materials themselves and "design[s] a system that sufficiently and effectively constrains the learners' activity in ways that press them toward correct conclusions" (Ball, 1991a, p. 9). In Ball's fourth approach, students are provided with structured materials, but they are expected to make and discuss their own constructions with the materials, thus dealing more directly with the various ambiguities inherent in any mathematical representation. The teacher in this fourth approach, rather than presenting particular ways to use and talk about particular representations, serves as a guide and support in students' efforts to figure out powerful ways to work with the materials and mathematical ideas.

Hugo's use of manipulatives is most like Ball's third approach. She carefully plans structured activities with manipulatives or other representations that will help lead students to particular understandings. She establishes conventions for how manipulatives are to be used and talked about and shows students how to use them, making explicit the links between various actions on the manipulatives and the mathematical concepts, written symbols, and procedures being taught. For example, in the showing decimals lesson, Hugo was careful to specify what each kind of base-ten block would represent and to establish rules for how to talk about them and place them on the place-value mats.

A number of scholars writing about mathematics learning and instruction have argued for this sort of careful designing of instructional representations and activities. For example, Nesher (1989) has argued that, in developing learning activities or curriculum, one should begin with the mathematical ideas to be learned, then carefully design a system of representations and accompanying rules and language for working with the representation. Working within this learning system (the combination of representations and rules), students will be able to construct correct and powerful mathematical concepts without them having to rely solely on the teacher to make judgments about what is mathematically correct. Fuson and Briars (1990) similarly argue for an instructional approach that leads the child to "construct the necessary meanings by using . . . a physical embodiment that can direct their attention to

crucial meanings and help constrain their actions with the embodiments to those consistent with the mathematical features of the systems" (pp. 181-182). The desirability of constraining learners' activities with manipulatives and other representations to lead them to particular understandings is also implicit in Resnick's (1987) call for instructional theory

> that helps us to understand how specific kinds of explanations and demonstrations are used in the knowledge construction process, what forms of practice will encourage the invention of correct and powerful procedures, and what kinds of feedback will optimally guide knowledge construction. (p. 47)

In all these cases, researchers are calling for the careful design of representations and accompanying language and activities to help students acquire particular mathematical understandings.

Carefully structured use of concrete materials and other representations also figures prominently in successful Asian mathematics instruction, as described by Stevenson and Stigler (1992; Stigler & Stevenson, 1991). In their comparative studies of elementary-school mathematics classrooms in Japan, Taiwan, and the United States, Stevenson and Stigler have found that Asian teachers are more likely than American teachers to use concrete materials to represent mathematical ideas and to use the same representations repeatedly and systematically, rather than drawing on an arbitrary variety of different representations. "Through the skillful use of concrete objects, Asian teachers are able to teach elementary school children to understand and solve problems that are not introduced in American curricula until much later" (Stevenson & Stigler, 1992, p. 187). Hugo shares with these approaches an emphasis on using concrete materials and other representations within carefully structured activities to lead students to particular mathematical understandings.

Through her talk to us about representations she was considering, Hugo revealed considerable knowledge of the mathematics she teaches and about various representations she might use, an important component of pedagogical content knowledge as originally defined by Shulman (1986a). In part because of her role as district mathematics coordinator, Hugo was familiar with a number of curriculum materials and activities, particularly ways to teach

various mathematical concepts through the use of problems and manipulative materials. She talked, for example about the Middle Grades Mathematics Projects materials (she used The Mouse and the Elephant [Shroyer & Fitzgerald, 1986] from this series during the year) and various NCTM books and videotapes on using manipulatives. Hugo also considered carefully the various representations she chose to use--about how well they represented the mathematical ideas she wanted to teach and how they might relate to other materials she had been using. Hugo talked to us, for example, about her decision to use base-ten blocks across the year in different ways. She used them to teach multiplication and division and again to teach decimals. An example of the kind of thinking Hugo did about instructional representations is provided in the following excerpt from her stimulated-recall interview in which she is talking about having students represent various decimals like .8 with base-ten blocks:

> I'm still trying to decide if it's confusing for the kids or not. [I] was thinking of using the base-ten blocks tomorrow for division but don't want to go back and forth in the middle of learning decimals [i.e., not using base-ten blocks in two different ways]. It seems to be going okay for most of the kids in terms of making the switch--that now instead of 100 this is one whole [flat].

Verbalization and Writing

Along with structured use of manipulatives and other representations, verbalization and writing play important roles in Hugo's teaching, both how she talks about it and what she does in the classroom. The importance of discourse--how teachers and students talk and communicate about the ideas they are learning--has received considerable attention in much of the rhetoric about reforming mathematics teaching and learning in the United States. "Mathematics as communication" is one of the standards that cuts across all levels of NCTM's (1989) Curriculum Standards, described as follows:

> Young children learn language through verbal communication; it is important, therefore, to provide opportunities for them to "talk mathematics." Interacting with classmates helps children construct knowledge, learn other ways to think about ideas, and clarify their own thinking. Writing about mathematics, such as describing how a problem was solved, also helps students clarify their thinking and develop deeper understanding. (p. 26)
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> Middle school students should have many opportunities to use language to communicate their mathematical ideas. . . . Opportunities to explain, conjecture,

> and defend one's ideas orally and in writing can stimulate deeper understandings of concepts and principles. . . . Unless students frequently and explicitly discuss relationships between concepts and symbols, they are likely to view symbols as disparate objects to be memorized. (p. 78)

Because of her interest in mathematics teaching and her role as mathematics coordinator, Hugo was familiar with these ideas,[5] as reflected in the following excerpt from her December interview, in which she responded to a question about her role as a teacher in helping students learn through problem solving. She also commented on the way CSMP, the district-adopted curriculum, had helped increase the useful talking about mathematics in many teachers' classrooms:

> Provide the problems for them. Provide some direction for them. Give them some time to think about it themselves whether it is individually, small groups, or with their parents. Come back and talk about it. Give them some more time to work on some things, come back and talk about it. Verbalization I think is one of the neat things with CSMP and it is not just in CSMP. Again, depending on the teacher, but CSMP has helped a lot of teachers do more verbalizing in math whether it's just by asking more questions, but getting the kids to talk more about mathematics.

For Hugo, verbalizing through talking and writing is important for keeping students actively involved in instructional activities and for providing evidence that students understand what they are learning:

> Like it goes with the old idea that it gets the kids being more actively involved. If the kids are just sitting there and not doing anything but they are being talked at by the teacher they are not going to catch as much as if they're the ones that are actually solving the problems, figuring it out. If they kind of construct their own--well like the theorems that I have up on the front of the room right now. . . . If the kids can explain it to me and--like Janice's theorem right now, which is basically how to find the perimeter of something. But one of the students was able to verbalize it for me and relating back that way. I mean she is probably always going to remember it. Hopefully, the rest of the kids are able to verbalize it themselves to you before we finish up with everything instead of just saying, "To find the perimeter you are going to take the length times the width and double it." If I can get them to tell me that, it is going to have a lot more meaning for them.

Verbalization--providing opportunities for students to talk and write about mathematics--figured prominently in three aspects of Hugo's teaching: (a) her use of

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[5]Similar ideas about discourse also figure prominently in NCTM's (1991) Professional Standards for Teaching Mathematics, but this document had not been released the year we worked with Hugo.

cooperative learning techniques to provide opportunities for student discussion, (b) her use of writing in mathematics, and (c) the discourse in her classroom.

Cooperative learning. Hugo's mathematics lessons often involved having students work in cooperative groups of four or five, which she referred to as "learning buddies." In interviews, Hugo talked about using these groups to provide opportunities for students to work together and to talk about mathematics. Hugo had in the previous year attended district inservice workshops on cooperative learning approaches that emphasized the importance of helping kids learn to work effectively in groups by providing a structure and assigning roles for individuals in the groups.

In setting up small-group activities and talking to students about their group work, Hugo emphasized what students should be doing in the groups and that the goal of their group work was for everyone in the group to learn and understand, not just to complete a worksheet or assignment. Hugo typically introduced a small-group assignment by writing the task to be completed on the overhead or blackboard and making a list of the various roles, or jobs, that each group would need to assign.

For example, during the showing decimals lesson, Hugo gave learning-buddy groups a worksheet on which they were to use base-ten blocks to represent a variety of decimal numbers. She pointed out that the students' job in groups is to teach other people and that "I need these worked for understanding. Make sure you have understanding by everyone in the group. It's okay if you're not finished as long as you're working." She then listed on the board the group roles that would be needed--recorder, timekeeper, and so forth. She said that one person should be moving the blocks, one person writing on the sheet, and everybody discussing. Only when everyone agreed on an answer should the recorder write it down. All group members were to sign the sheet to indicate that they understand everything done on the paper.

As Hugo had students break up into their learning-buddy groups, she told them they would have 15 minutes and that she needed their sheets turned in at 9:50. As 9:50 approached, students from the groups brought their sheets to Hugo and returned to their seats. Hugo then

began a whole-class discussion of what went on in the learning-buddy groups. She complimented various groups for working well together and asked for suggestions for how various groups successfully made assignments about who would take each role (timekeeper, etc.). One group had apparently had difficulty making these assignments, so Hugo asked the class for suggestions for how to get the roles assigned. Hugo then talked briefly about two problems on the sheet that some of the groups had difficulty with. She made more comments about group process--about the need for good eye contact, good communication back and forth--and that, "hopefully, signatures mean you're understanding."

The emphasis here on how students should work together in groups was typical of Hugo's talk about groups. She usually began and ended small-group time with comments or discussion about expected behavior in the groups, asking students for suggestions for constructive group behaviors. She sometimes monitored learning-buddy groups by moving about the room making notes about positive and negative group behaviors on a clipboard. After learning-buddy groups one day, for example, Hugo said she saw most groups "on the same level" (sitting in chairs or on the table, with no one physically higher or lower than other group members), working "close" (no stray members), and lots of good eye contact. Hugo frequently pointed out that the goal of group work was to help each other learn and to understand the material.

When problems arose, Hugo had students discuss ways to work together. For example, after one small-group activity, Hugo told students she had heard good questions and good discussions. She wanted to know, however, why one student did not want to share an answer. The student responded by saying she did not want anybody to copy. Hugo said that it made her think that this student had gotten some embarrassing comments about her answers. She said it was important for students to work as a group and that they would not change groups until they learned to work together. She mentioned a word that the students might want to use in the future--consensus. She asked the students what it meant and received replies such as "an idea" and "agree on something." After class, we asked Hugo what she would consider "good discussion"

in a group to be. Hugo said it was the things she had said in class: "Working together, being kind, challenging answers politely, and so forth."

This emphasis on roles and process in small groups is consistent with recommendations in much of the cooperative learning literature (e.g., Slavin, 1983), which points out that students need to learn the new roles and behaviors for working together and that the organizational structure of the classroom must provide adequate incentives for students to stay on task and help one another learn. These aspects of cooperative learning are important; failing to attend seriously to them can wreak havoc on any classroom's order and management, creating an environment in which little learning can take place. But just as an emphasis on the effective generic teaching behaviors of process-product research can result in a lack of attention to the nature of the subject-matter content being taught (Shulman, 1986b), an exclusive focus on the form and process in cooperative groups can result in significant omissions. By focusing so intently on the roles and behaviors of group members in learning-buddy groups, Hugo seemed at times to ignore the mathematics students were supposed to be learning. Seldom was there any discussion of the mathematical problems or solutions that arose during learning-buddy groups.

There is a bit of a paradox here. On the one hand, Hugo talked to students regularly and emphatically about the importance of understanding what they were working on together in learning-buddy groups. On the other hand, because virtually all of the class talk about work in groups revolved around roles and general behaviors for engaging in conversations with peers (unquestionably important aspects of group work), students seldom participated in or saw modeled meaningful give-and-take discussions about mathematics.

Writing. Hugo often had students write about mathematics, usually having them explain in writing some mathematical idea or procedure. In one assignment, for example, students were to explain to a five-year-old how to read the number 0.87 and what it means. In another assignment, students were to explain in writing what $\frac{2}{3} \times 24$ means. Hugo talked about using these written explanations to help her find out if students "really understand" and are not

just "crunching numbers." "If they can put in writing or put in words each step then I would feel like they really do understand multiplication and that they're not just memorizing rules and lucky enough to have remembered them." The written explanations served in place of face-to-face interviews with students, which would be too time-consuming to conduct on a regular basis. Hugo also spoke about a current "movement" that emphasizes writing as a way to promote understanding. The movement is based on the notion that putting something into words verbally or in writing will force you to think about it.

As with her use of manipulatives, when Hugo had students explain through writing, she usually had some image of an "ideal" explanation in mind--a fairly clear notion of the expected form of a good explanation. When students turned in written explanations that failed to reach this standard, Hugo had the individuals or groups responsible for them rework them. She sometimes used examples of explanations that students had written to show students what she considered to be a good explanation. For example, after students had turned in written explanations for $\frac{2}{3} \times 24$, Hugo photocopied three student explanations to discuss with the class (Figure 2). The class had been talking about fractions in terms of a visual representation and accompanying story from the CSMP curriculum about a gorilla named Bobo, who ate various numbers of bananas. The following fieldnote excerpt describes how Hugo drew attention to the strengths and weaknesses in these explanations written by students:

> Hugo passed out the sheet of explanations to the class, asking students to make comments on the different explanations, stating, "I don't want to know whose explanation it is, only what you think of it." She said she was pleased by the class's explanations as a whole but wanted to point out some examples of confusing words. She started by pointing out that she believed the student who wrote the first example could do the algorithmic manipulation. She said: "Look at the first example. I have a feeling that this person understands how to use the numbers, but from this paper I'm not sure that this person really understands what's happening. I can give this person some numbers and they can come up with the right answers. But what I want to make sure of is that each of you really understands what it means to take two-thirds of 24." Hugo then turned attention to the second example. She pointed out that the picture was accurate and then asked the class if they were splitting 24 into three parts. A few students responded with a soft "yes." Hugo then asked what the explanation was missing. When there were no responses she asked "What should go between 'three' and 'parts'?" A few students responded with "equal." Hugo then explained that

there are important parts that need to be included in the explanations, and that splitting 24 into three equal parts was one of the important parts. Hugo highlighted the third example as the best explanation, because it was "nice and concise, compact"; it had most of the important parts; and it was easy to understand.

This brief episode illustrates the emphasis Hugo places on learning to understand, not just memorizing the steps of computational procedures. In this lesson, as in many, Hugo told students explicitly that she expects them to understand the mathematics they are learning. Implicit is the idea that writing is a way to demonstrate one's understanding, an idea that Hugo commented on in a brief interview after this lesson. We asked Hugo if it is important that students be able to make their understandings explicit and she responded "Yes, if the student[s] can explain it clearly they're going to be able to transfer that knowledge into other areas."

It is also apparent in this episode that Hugo has fairly clear expectations about the form and elements of a good explanation. She has a "correct" and complete explanation in mind and expects students' explanations to match that target, just as she has particular understandings that she hopes students to arrive at when working with manipulatives and other representations. In the interview after this lesson, Hugo said that how much she expects from any individual student depends on their "developmental level," but that there is ultimately a particular set of important parts to a good explanation to be held up as a goal. By presenting these three examples of written explanations, Hugo conveyed what she expected in a good explanation and was explicit about what was lacking in the others. Thus there was sense of convergence in what constitutes a good explanation or "understanding." Like the Japanese teachers described by Stevenson and Stigler (1992), Hugo used students' "errors"--in this case faulty written explanations--as an important part of the lesson. Like the Japanese teachers, Hugo used faulty student work by explaining what was wrong with it, contrasting it with the "correct" or complete explanation.

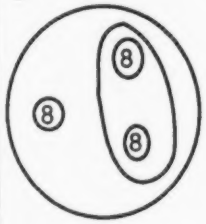
Classroom discourse. Hugo's use of cooperative groups and having students write about mathematics were highly visible techniques she was incorporating into her teaching to

1a.

$\frac{2}{3}$ x 24 You take the 3 and + in to 24 thats were you use the 3 the you times the answer to that 2 then you end up with the answer.

1b.

$\frac{2}{3}$ x 24 = 16



The reason I put 8 in each group is I split 24 in to 3 parts and Bobo ate two parts and 8 + 8 = 16 so Bobo ate 16

1c.

$\frac{2}{3}$ x 24 means you take 24 and divide it into three equal groups. There are 8 in each group. Then you take 2 of those 3 groups and combine them for your answer.

Figure 2. Three student explanations of $\frac{2}{3}$ x 24=16.

create opportunities for students to verbalize more about mathematics. In the discourse of her mathematics classroom--the ways that Hugo and her students communicated with one another about mathematics--lay more subtle but significant aspects of both change and continuity in her teaching. As argued in NCTM's (1991) Professional Standards for Teaching Mathematics, "The discourse of a classroom--the ways of representing, thinking, talking, agreeing and disagreeing--is central to what students learn about mathematics as a domain of human inquiry" (p. 34).

In this section we consider some of the ways Hugo communicated through classroom discourse her emphasis on understanding mathematics. We also look at some of the interactions that took place during whole-class lessons, considering the sorts of opportunities Hugo was trying to provide for her students to participate in richer and more powerful mathematical discourse.

Hugo spoke often and explicitly to her students about the importance of understanding what they were learning. For example, in the showing decimals lesson, she opened a discussion of the similarities between decimals and fractions by writing the words "decimal" and "fraction" on the board, saying, "Tell me some things that are similar about those words up there. Not the words themselves, but the ideas behind them." A bit later in the same lesson, Hugo said when introducing the small-group task, "I need these worked for understanding," and pointed out that everyone in the group should understand what the group has done before they were finished. These explicit statements about understanding were corroborated by comments Hugo made during the stimulated-recall interview for the lesson. For example, referring to having students show .8 with base-ten blocks (the first task in the lesson), Hugo said she was trying to see "who's making sense of eight-tenths and isn't at that point."

Hugo's emphasis on understanding also showed up through the number of important mathematical ideas that came up during lessons as a result of the tasks she posed and the questions she asked. During the showing decimals lesson, for example, the following ideas related to decimals emerged during whole-class discussion: (a) equivalence of ten-tenths to

one whole; (b) equivalence of one-tenth to ten-hundredths; (c) discussion of whether 2 is a decimal, a fraction, or both; (d) whether one can have negative decimals (a question raised by a student) and, if so, how to represent them with base-ten blocks; (e) whether all decimals can be expressed as fractions and vice versa; (f) with fractions, thinking of a shaded circle as $\frac{1}{8}$ or $\frac{7}{8}$, depending on whether you are referring to the shaded or unshaded portion; (g) with decimals, the whole is always divided into 10 parts, 100 parts, and so forth, which are "powers of ten"; (h) and the difference between 5 tenths and .5 tenths. The fact that such ideas even surfaced during this mathematics lesson is evidence of Hugo's moving beyond the exclusive emphasis on carrying out the steps of computational algorithms decried by mathematics educators.

In addition to communicating an emphasis on understanding, Hugo worked to incorporate into her teaching pedagogical strategies for increasing students' opportunities to talk and reason about mathematics. When the Curriculum Standards (NCTM, 1989) authors refer to the importance of students having opportunities to talk about and interact around mathematics, they emphasize the importance of students having "opportunities to explain, conjecture, and defend one's ideas orally" (p. 78). Hugo had clearly picked up on this language in talking about what she does in the classroom and was attempting to incorporate these ideas into the discourse of her classroom.

Hugo spoke often, both to us in interviews and to students during lessons, about the importance of explaining things verbally. Indeed, it is partly on the basis of these statements that we made our claim that explaining is central in Hugo's definition of understanding. An example of Hugo's talk about explaining in interviews comes from a lesson dealing with the idea of equivalent fractions. After the lesson we asked Hugo if it was important to her that students be able to state or talk about the equivalence or if understanding it intuitively was enough. Hugo said she thought it was important for students to be able to talk about it, to explain why

explicitly. If they really understand, she said, if they are to be able to use their knowledge in the future, to go to something else, they need to be able to verbalize it.

In another lesson, the class was checking homework that involved two-digit multiplication computation. Hugo had students talk through the steps of each problem, explaining why each step was taken and what the "real" value of each number was (e.g., a 2 in the tens column is "really" 20). After talking through some of these problems, Hugo emphasized to students that it is not enough to be able to do the computation; she expects them to understand--to be able to explain:

> Now you may think, "I had that answer, we didn't need to go through all of this." However, what if I gave you a problem like 24 times 36 and I said, "Explain to me in writing, everything that you do." You know that's my kind of question don't you? . . . I want you to be able to explain it. As we started off some of you said, "Well, we do it because that's how we do it." I want you to know more than that. I want you to know we do it because it has a value of 20. We do it because the 2 goes up because it's 2 groups of 10 that we're gonna carry.

As with written explanations, there was often a sense of convergence behind Hugo's frequent requests for verbal explanations; students were expected to explain things in correct and particular ways. But juxtaposed with this convergence were clear attempts by Hugo to provide opportunities for students to offer and justify alternative answers and solutions, as advocated by the Curriculum Standards (NCTM, 1989).

For example, in a lesson in which the class was using the "Bobo" representation from CSMP (CEMREL, 1985) to represent multiplication of a whole number by a fraction, Hugo had displayed the diagram in Figure 3 on the overhead (with no numbers filled in). She had told the class there were 20 bananas to be divided up among five monkeys and a student had said there should be 4 bananas for each monkey. Hugo had written "20" in the box to show the total number of bananas and "4" in each of the ovals to show how many bananas each monkey got. Hugo reminded students that the red line (dotted line in the figure) showed how many bananas Bobo ate and asked what "three monkeys worth of bananas" would be--"the part inside the red." Three students responded, Wendy saying "three-fifths," John saying "three-fourths," and Dov saying "12 bananas."

Hugo responded to Dov by saying that he was right but that she wanted an answer at this point in terms of the fraction of bananas. Hugo then said, "We have three-fifths and three-fourths. Convince me which one is right and why." (Note here that Hugo was explicitly providing an opportunity for students to justify or defend conjectures--in the language of the Curriculum Standards, to "discuss mathematical ideas and make conjectures and convincing arguments" [NCTM, 1989, p. 78]). Rob said the correct fraction was three-fifths "because there are three of them [ovals or groups] in the circle [the red enclosure] and there are five [groups] in all." Ben said it was three-fourths because Bobo "ate three groups of four." There was a bit more discussion, with no new arguments put forth, when Hugo said, "We need to agree about what we will call this [pointing to the enclosed three groups]. Let's agree, and mathematicians agree, to call this three-fifths." She then went on to say, " Ben has an important point and we need to decide what to call the four bananas." Hugo asked the class, "What are the four bananas?" A student responded that four bananas are one group, which is one-fifth of the bananas. Hugo concluded by saying, " Ben, one-fifth is four bananas," writing $\frac{1}{5}$ on the overhead.

This episode again illustrates Hugo's emphasis on having students understand the mathematics they are learning, in this case by reasoning about multiplication of fractions in the context of a visual representation. Hugo also provided opportunities for students to offer alternative conjectures and justify them. Once these student ideas were expressed, Hugo emphasized the correct responses, offering explanations to counter incorrect answers or reasoning. As often happened when we observed in Hugo's class, we were not sure whether students "got" the explanations that Hugo offered--whether they were making sense of what she had said. Hugo was apparently satisfied with the explanation to Ben that four bananas were one-fifth of the total, but Ben had no opportunity to say whether that made sense to him or whether he was able to connect that explanation to the way he was thinking earlier--three-fourths,

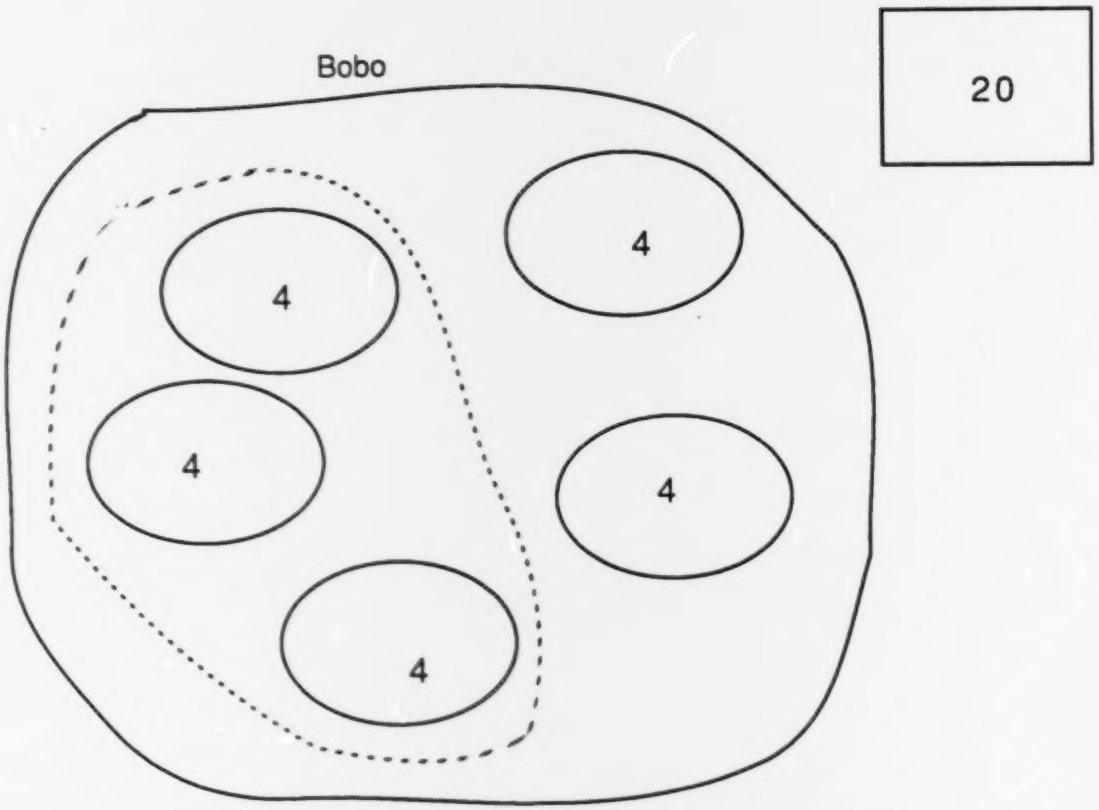


Figure 3. The "Bobo" representation from CSMP (CEMREL, 1985).

meaning three groups of four bananas. Furthermore, it was not clear that John's reasoning in his original answer of "three-fourths" was the one reflected in Ben's justification.

This incongruity--on one hand eliciting students' thinking, and on the other hand cutting short students' thinking by pressing for convergence and offering explanations--reflects a fundamental tension inherent in teaching for understanding (Ball, in press; Lampert, 1985; Putnam, 1992b). Learning with understanding requires that students somehow make sense of what they encounter. In the language of cognitive psychology, students must integrate new information and experiences into their existing cognitive structures, and it is only through these existing knowledge and beliefs that they can interpret and make sense of instructional events. But in facilitating students' efforts to make sense, the teacher must somehow ensure that students come to generally accepted or "correct" understandings. The teacher is expected to teach particular mathematical content and help students learn particular mathematical ideas and skills. Traditional images of teaching, with their emphasis on modeling and direct explanation, tend to emphasize the "correct understandings" side of this tension, with the implicit assumption that if students are exposed to and rehearse correct knowledge they will learn it. In contrast, much of the recent rhetoric of reform in mathematics education emphasizes the constructed nature of understanding (e.g., NCTM, 1989).

Recommendations such as having students conjecture and justify mathematical ideas are based in part on the notion that students must actively construct their understandings through thoughtful interactions with others.[6] Teachers are faced with the dilemma of trying to provide opportunities for students to build richly on their own thinking and efforts after meaning while still ensuring that students learn certain correct and powerful mathematics and the class as a whole covers expected content. Hugo seemed to deal with this tension by offering students opportunities to express diverse solutions and ideas, but then dismissing or exposing erroneous

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[6]The current emphasis on mathematical reasoning, argument, conjecturing, and justifying also has roots in the belief that these are important aspects of mathematical thinking that have been largely ignored in school mathematics (Putnam, Lampert, & Peterson, 1990).

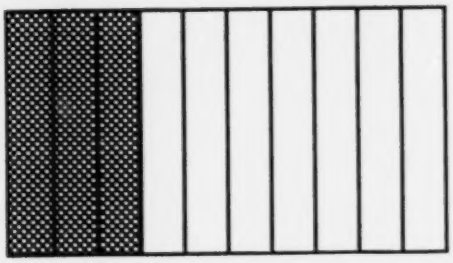
thinking quickly and focusing on the "correct" explanations. The opportunities for students to contribute were superimposed on the backdrop of Hugo's existing patterns of interacting with students and deeply held and rich expectations of the mathematics she wanted students to learn. The result was that many important mathematical ideas were raised and valid explanations were offered by Hugo and her students, but these explanations sometimes sped by. It was not clear how many students were making sense of the explanations.

Student Learning

As part of our observation in Hugo's classroom, we videotaped and observed lessons for two weeks during a unit on decimal fractions. To get a sense for what students learned during this unit we conducted interviews before and after the unit with six target students, three boys and three girls Hugo had identified as low-, middle-, and high-achieving students in mathematics. The interviews included a variety of tasks involving explaining, ordering, and adding decimal numbers. Our goal was to assess what knowledge of decimals the students brought to the unit and what they learned. In this section, we briefly characterize the performance of the six target students on these interviews before and after the unit.

In both interviews, we showed students the number, .7, and asked how they would say that number. In the preinterview, all six target students said the number would be read "point seven." Only two students, John and Melody, wrote the number as a fraction ($\frac{7}{10}$) when asked to write it another way, and only John could give an explanation of what the number meant, saying it was like 70 cents and drawing a rectangle with 7 out of 10 parts shaded. In the postinterview all the students said this number was "seven-tenths" and all but Janet could talk about the number meaning 7 out of 10 parts or draw a picture to show 7 parts out of 10.

To see whether students could interpret a simple part-whole representation as a fraction or a decimal, we asked students to write a number represented by the following diagram, which was similar to a visual representation used by Hugo during the unit:



We also asked students if they could write the number any other way. Before the decimals unit, all target students except Richard (who wrote 3.7) wrote $\frac{3}{10}$ as representing the diagram.

Melody, John, and Rob also wrote the number as .3. At the end of the year, all six students wrote both .3 and $\frac{3}{10}$.

Another task involved identifying the larger in each of the following pairs of numbers:

4.8, 4.63	4.7, 4.08	4.4502; 4.45	0.5, 0.36
2.621, 2.0687986	0.25, 0.100	$\frac{4}{100}$, 0.038	0.457, $\frac{4}{10}$

The students in general had difficulty identifying the larger fraction, with their reasoning apparently following two implicit faulty rules for ordering decimals (Resnick et al., 1989). One group of students used what Resnick et al. called a whole number rule in which the digits to the right of the decimal are essentially treated as though they formed whole numbers. Thus, a number with more digits to the right of the decimal point is identified as larger than a number with fewer digits to the right of the decimal point. For example, 2.0687986 is identified as larger than 2.621 and 0.36 is identified as larger than 0.5.

The other common rule was virtually the opposite, what Resnick et al. (1989) called the fractions rule. Students judged the number with the smallest number of digits to the right of the decimal point as the larger of the two. Resnick et al. suggested that students following this rule have generalized the idea that the size of the pieces gets smaller with each place to the right--that is, thousandths are smaller than hundredths--so any number extended to the thousandths place is smaller than a number extending only to the hundredths place. In the postinterview, Melody and John, who had difficulty in this section in the preinterview

consistently identified the larger decimals correctly. The other four target students, however, continued to use faulty rules.

At one point in the interview we presented three addition computation problems that involved decimal numbers of various sizes:

35.6 + .145 =
326.27 + 3.007 =
309.78 + 94.064 =

In the preinterview two of the six target students (John and Melody) carried out the computation correctly, writing the numbers in vertical format with the decimal points aligned before adding. Neither John nor Melody, however, could explain why the numbers needed to be lined up before adding. Both of these students provided a conceptual explanation of why the numbers needed to be aligned in the end-of-year interview, talking about the need to keep tenths together with tenths, hundredths with hundredths, and so forth. In the preinterview, the other four students wrote the numbers aligned on the right-hand side as if there were no decimal point and computed the problem using the traditional addition algorithm, using a variety of methods for deciding where to place the decimal point in the sum.

In the postinterview, two of these students, Janet and Nancy, again aligned the numbers on the right and added. Richard wrote the numbers with the decimal points correctly aligned, but added the portions to the right of the decimal points inconsistently, sometimes treating them as whole numbers. Rob added the numbers mentally, first adding the whole-number part of the numbers, then the decimal part. In adding the decimal parts, however, he ignored the zeros and thus produced incorrect sums on all three problems. For example, when adding 326.27 + 3.007, Rob added 326 and 3 to get 329. Then he added .27 and .007 as 27 and 7 to get 34, producing the sum, 329.34. Thus, only John and Melody, who were already carrying out the addition algorithm correctly in the preinterview, were able to add decimals correctly in the postinterview, although now they could explain the procedure.

In the final task of the interview, the students were asked to order a set of cards displaying decimal numbers, fractions, and whole numbers:

.50, .5, $\frac{1}{2}$, $\frac{3}{6}$, .25, $\frac{1}{4}$, $\frac{1}{10}$, .1, $\frac{1}{100}$, .01, .43, .61, .009,
.023, 1, 10^1, 10^{-2}, 10, $\frac{1}{5}$, .2, $\frac{1}{3}$, .474, .6, .044

This proved a difficult task for the students, both in the pre- and postinterviews. None of the target students ordered the numbers correctly in the preinterview. They appeared to use various incorrect rules to order the numbers or they placed the cards arbitrarily. On the postinterview John and Melody ordered the numbers for the most part correctly, with a few misplaced or omitted numbers (none of the students were able to interpret the numbers with exponents). John ordered the cards as follows (numbers in the same cell were grouped as equal):

$\frac{1}{100}$ .01	.009	.023	.044	$\frac{1}{10}$ .1 10^1	.2 $\frac{1}{5}$ 10^{-2}	$\frac{1}{4}$ .25	$\frac{1}{3}$	.43	.474	$\frac{3}{6}$ $\frac{1}{2}$ .5 .50	.6	.61	1	10

Melody's ordering was:

.009	.01 $\frac{1}{100}$	.023	.044	.1	.2	.25 $\frac{1}{4}$	.43	.474	.5 .50	.6	.61	1	10

In general, only John and Melody showed much improvement on our interview tasks. Both had been identified by Hugo as strong students and both showed some knowledge of decimals, especially computational algorithms, before the unit. After the unit on decimals, these two students were better able to identify the larger of two decimal numbers; identify, draw a picture of, and explain a decimal number; write a decimal and fraction that corresponded to a rectangular representation of the number; correctly line up like parts and add decimal numbers; and order decimals, fractions and whole numbers. In short, their conceptual and

procedural knowledge of decimals had grown. The other four students, Janet, Nancy, Richard, and Rob, performed after instruction much as they did before. Aside from improvement in correctly identifying a shaded rectangle as .3 and knowledge that .7 means 7 out of 10 parts, these students showed little improvement. They continued to be unsuccessful at choosing the larger of two decimals or making a coherent ordering of the set of numbers. The errors these students made in the postinterview were often the same errors they made in the preinterview.

Hugo clearly wants her students to learn important mathematics and to learn it with understanding. She wants them to be able to explain the various procedures they are learning and to know when and how to use them. We have seen that some of the students in Hugo's classroom were learning what she wanted them to learn. John and Melody had significant misconceptions that were corrected during the decimals unit and they were able to give more conceptual explanations about decimals of the sort that Hugo expected. But other students seemed not to be learning in the way that Hugo had hoped. A number of the target students showed little improvement in knowing the mechanics or the conceptual basis of decimals.

Conclusion

What can we learn from this portrait of Hugo's efforts to teach mathematics for understanding? First, teaching for understanding is uncertain. Taking on the challenge, as Hugo has, of helping students understand the mathematics they are learning introduces many uncertainties into the already complex task of teaching. Some of this uncertainty concerns just what it is students should be learning—what it means to understand. When the instructional goal is ensuring that students are able to carry out a collection of computational algorithms accurately, the desired outcomes are clear (being able to carry out the steps of computational algorithms). When the goal is understanding, it is not so clear just what and how students should be learning or what to take as evidence that students have learned. The teaching and research communities hold forth multiple views about what mathematical understanding is (Putnam et al., 1990). For some, understanding means having knowledge situated in real-

world contexts (Brown, Collins, & Duguid, 1989), for others it means having richly linked and interconnected mental structures (e.g., Hiebert & Carpenter, 1992), whereas for others it means being able to engage in mathematical conjecture and argument. Reform documents like the NCTM (1989) Curriculum Standards incorporate all of these ideas about understanding, and more.

That is a great deal--maybe too much--to think about while planning instruction and interacting with students on a moment-to-moment basis. To teach mathematics for understanding requires having a definition, even if implicit and inconsistent, about what it means to understand. Whereas a teacher needs some definition, at least implicit, of what it means to understand mathematics to guide decisions, this definition is bound to be less explicitly and completely thought out than the views expressed in the scholarly literature. Unlike university researchers, teachers usually do not have the luxury of time to reflect extensively on whether their thinking about understanding is complete and consistent. Teachers must focus their ideas in ways that enable them to act in the classroom. Hugo seemed to manage the complexity of what it means to understand by focusing on the importance of explaining what mathematical symbols mean and why computational algorithms work. She expressed this emphasis explicitly through her statements about goals for students and implicitly through her instruction.

A second kind of uncertainty in teaching for understanding concerns what instructional techniques to use. Teaching for understanding is not a matter of implementing some instructional package or set of pedagogical techniques "off the shelf." Any teacher must draw on a wide array of materials, information, and recommendations to forge a teaching practice within the context of his or her classroom. This unique teaching practice must make sense to the teacher and must fit the numerous constraints that bear on classroom teaching, including the knowledge and beliefs the teacher and individual students bring to the classroom, parental expectations, district curriculum, expectations and pressures from other teachers. One particularly salient constraint in Hugo's case was the pressure to prepare her fifth-grade

students for middle-school mathematics teachers who would judge and place students in classes on the basis of their computational skills.

Forging a coherent and meaningful teaching practice can be especially complex for a teacher like Hugo, who actively seeks out new ideas and approaches in her efforts to improve. There is no shortage of people with recommendations for what teachers should do differently, no dearth of ideas and materials for teachers to consider. Hugo, as a professional, worked hard to integrate many new ideas that she read about and encountered in inservice activities and interactions with colleagues, including cooperative learning techniques, "effective teaching" behaviors such as specifying objectives clearly for students, systematic discipline strategies, use of manipulatives to embody mathematical ideas, having students write, and having students verbalize and justify their reasoning. But all these good ideas do not come in a ready-made package. Such ideas or recommendations typically deal with some particular aspect of instruction and are sometimes based on diverse or even competing assumptions about good teaching. The teacher must selectively choose to incorporate these ideas or not into his or her classroom instruction. Teaching is by its very nature eclectic.

An important and related point illustrated by Hugo's teaching is that a teachers' eclecticism--his or her incorporating of ideas, materials, and techniques from multiple sources--is filtered and shaped by that teacher's existing knowledge and beliefs (Borko & Putnam, in press; Putnam, Heaton, Prawat, & Remillard, 1992). Just as students' learning is shaped largely by their existing knowledge, teachers interpret and adapt information and ideas through what they already know and believe. When Hugo drew on new curricular materials, or ideas about cooperative learning or having students write, she shaped and adapted these ideas into her existing practice and her current ways of thinking about teaching, learning, and mathematical understanding.

For example, when Hugo offered students opportunities to write about mathematics, she structured the activities to require students to offer explanations of procedures or ideas. Consistent with her focus on having students develop particular conceptual understandings, Hugo

used these writing exercises to emphasize the importance of students ultimately being able to produce correct and clear explanations. A teacher with a different perspective on understanding might have used student writing in mathematics more as a way for students to describe various ways to solve a particular problem, or to express their feelings about mathematics. Similarly, Hugo's attempts to offer students opportunities to verbalize and justify their thinking were shaped by her focus on complete and accurate explanations

Because Hugo has relatively strong and rich knowledge of the mathematics she teachers, her teaching also sheds light on the role of subject-matter knowledge in teaching for understanding. Numerous writers have raised concerns about elementary school teachers' limited knowledge of the mathematics (Ball, 1991b; Brown, Cooney, & Jones, 1990; Post, Harel, Behr, & Lesh, 1991). A growing number of case studies of mathematics teaching illustrate difficulties that teachers encounter because of limitations in their mathematical knowledge (Heaton, 1992; Putnam, 1992a; Stein, Baxter, & Leinhardt, 1990). In these classrooms, teachers presented incorrect solutions to problems, failed to detect errors in students' solutions, and overlooked central aspects of the mathematics they were teaching. These studies illustrate the importance of subject-matter knowledge by its absence--what can go wrong in lessons when teachers' knowledge is weak.

In contrast, Hugo had a secure conceptual understanding of the topics she was teaching, as well as familiarity with numerous instructional representations and approaches for teaching about particular topics. We did not see incorrect solutions left unaddressed in Hugo's classroom; Hugo invariably made sure that she or a student provided acceptable solutions or explanations before the class went on. There was ample and rich mathematical content in Hugo's lessons. As we pointed out earlier, however, it sometimes was not clear how accessible this content was to all students. Because of Hugo's emphasis on providing explanations, students may not have been making sense of the ideas in terms of their own current thinking. We argued that these apparently inadequate opportunities for students to make sense and to show their thinking arose from the way Hugo dealt with a fundamental tension in teaching for understanding: the

tension between the need for students to make sense of what they are learning in terms of what they already know and the need for the teacher to ensure the learning of particular and correct mathematics. Hugo tended to emphasize the correct content side of the tension, possibly at the expense of student sense making.

Hugo's teaching shows us that rich subject-matter knowledge alone is not enough for successful teaching for understanding. Hugo treats the mathematics she is teaching very seriously, and has a firm enough knowledge of the mathematics to ensure that the mathematics publicly discussed in lessons is correct and important. But without more extended and concerted attempts to help students connect their thinking with this important mathematics, Hugo's teaching seems to fall short of her goal of fostering rich conceptual understanding in all of her students. One might argue that this falling short is a result of Hugo's emphasis on correct and particular explanations--that such a convergent definition of understanding and approach to teaching is inherently unable to connect with students' thinking in ways that help them construct powerful understandings. Or one might argue that the failure of more students to develop the rich understandings Hugo was striving for was due, not to Hugo's convergent expectations, but to an inadequate following up on explanations to see if students had understood them, or to other features of her explanations (Leinhardt, 1987). Or one could argue that other factors, such as interruptions from outside of class, disrupted the coherence of lessons (see Stevenson & Stigler, 1992).

These are issues that simply cannot be resolved on the basis of a single teacher's efforts to teach mathematics for understanding. Hugo's efforts do highlight, however, what may be a danger inherent in approaching teaching for understanding by emphasizing particular representations and understandings--that when focusing on these issues, teachers should take great care to ensure that students are making sense. The analogue to this danger in teaching approaches that take as their starting point the sharing of students' solutions and reasoning is that a lot of diverse student thinking may be aired, but students may not come to more powerful mathematical understandings. These sorts of tensions and ambiguities are an integral part of

teaching for understanding and part of the reason such teaching cannot be prescribed by textbooks or foolproof teaching algorithms, requiring instead what Cohen (1989) has called adventurous teaching.

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